Journal of Applied Mathematics and Mechanics 72 (2008) 323–330

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Neutral stability of compression solitons in the bending of a non-linear elastic rod☆

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ARTICLE INFO

Article history:
Received 9 November 2006

ABSTRACT

The spectral stability of compression solitons in non-linear elastic rods with respect to perturbations of the flexural mode of the oscillations of the rod is investigated. The system of equations of the isotropic theory of elasticity, taking account of the non-linear corrections corresponding to the interaction being studied, is used to describe the interaction of longitudinal and flexural waves in the rod. This system of equations describes long longitudinal-flexural waves of small but finite amplitude. It is shown that trapped flexural modes exist, which propagate together with a compression soliton. It is established that these modes, which are the least stable, do not increase with time.

The problem of the stability of a compressed rod to flexural perturbations was considered by Euler when solving the problem of the least height of a column for which it may buckle under its own weight or an external load. It was found that a rod buckles under its own weight or, in other words, is unstable to small transverse perturbations, if its length exceeds an amount $2.80\sqrt{R/W}$, where the constant R represents the flexural stiffness of the rod and W is the weight of the rod. It follows from this that it is always possible to choose the length of a horizontal rod such that it is flexurally unstable for any specified compressive loads and, in the limit of an infinitely large extension, the rod is found to be unstable in the case of loads of an intensity which is as small as desired. This instability occurs as a result of an exponential growth of the flexural perturbations in the linear stage of their development.

By analogy with the static case, it is quite natural to pose the question of the stability conditions with respect to small transverse perturbations of moving domains of compression, in other words, compression waves in a rectilinear rod of infinite length, where these waves do not interact with the ends of the rod. In the case of a linear elastic rod, the solution of this problem is trivial, since the longitudinal and flexural modes do not interact. In the case of a non-linear elastic rod, this interaction, in the lowest order with respect to the strains, occurs due to the cubic corrections in the expression for the elastic potential.

The equation in the long wavelength limit, which takes account of the main non-linear effects as well as the dispersion due to the finiteness of the rod diameter, was apparently derived for the first time for longitudinal waves in an infinitely long rod by Ostrovskii and Sutin[1] (also, see Ref. 2). This equation describes non-linear waves in a reversible dispersive medium and belongs to the class of Boussinesq equations. By virtue of the fixed relation of the signs of the elastic constants, the solution of the equation is a compression soliton which corresponds to a solitary longitudinal wave which propagates along the rod without any change in form. This small amplitude soliton (due to the fact that only the main non-linear effects have been taken into account) is linearly stable to longitudinal perturbations.[3]

The question of the stability of longitudinal solitons to flexural perturbations is solved below, that is, a problem is considered which is analogous, in a known sense, to the static Euler stability problem. Unlike the Euler problem for an infinitely extended rod, the moving localized compression zones of constant form are found to be neutrally stable in any order with respect to the amplitude of the compression soliton.

The literature on the stability of solitary waves (and non-linear waves in general) in elastic media contains papers on the orbital stability of solitons in rods which are in a prestressed state,[4,5] of a loop-like soliton (Euler loops) in elastic rods which are inextensible to perturbations which do not evolve from the plane of a loop,[6,7] and of shear deformation solitons in a composite material.[8,9] Papers concerned with the formulation of the conditions for the instability of an Euler loop to perturbations emerging from the plane of the loop[10–12] as well as shear solitons in a composite material[13,14] are also on this theme.

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 3, pp. 466–476, 2008.

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doi:10.1016/j.jappmathmech.2008.07.011

Equations, describing two-mode wave interactions have been considered in the study of the non-linear resonance interactions of longitudinal and flexural waves in a thin elastic annulus,[15] plane wave motions in elastic rods in the case of linear longitudinal deformations and arbitrary flexures[16] and, also, three-wave interactions of longitudinal and flexural waves using the general Rayleigh model, taking account of the cubic terms in the equations for the flexural oscillations.[17]

Below, as a basis, we take the Rayleigh equations for the two-mode oscillations of a rod, where the interaction between the longitudinal and flexural deformations is taken into account using the cubic terms in the free energy expression.

1. Longitudinal waves

We shall consider an isotropic elastic rod. The elastic potential must depend on the components of the strain tensor $\varepsilon_{ij}(i,j = 1, 2, 3)$ through its scalar invariants

$$I_1 = \varepsilon_{ii}, \quad I_2 = \varepsilon_{ij}\varepsilon_{ij}, \quad I_3 = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}$$

(summation over repeated indices is assumed) In order to take account of non-linearity in the main order, the expansion of the free energy in powers of the components of the strain tensor is restricted to terms of up to the third order inclusive:[18,19]

$$\Phi = \lambda I_1^2/2 + \mu I_2 + AI_3/3 + BI_1I_2 + CI_1^3/3 \tag{1.1}$$

where λ and μ are Lamé coefficients, and A, B and C are third order elastic moduli. Experimental information is available concerning the measurement of these.[20]

It is assumed that the axial and transverse deformations are connected by the relation[21,22]

$$\varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11} \tag{1.2}$$

where υ is Poisson's ratio. The longitudinal displacement of the points of the middle line of the rod is denoted by $u = u(x,t)$, and $\tau_1 = \partial_x u(\partial_x = \partial/\partial x, \partial_{xx}^2 = \partial^2/\partial x^2$ etc.). Then, the components of the transverse shifts in the highest order with respect to the gradients of the displacements (for waves with a characteristic length which greatly exceeds the transverse dimensions of the rod) have the form $\upsilon y\tau_1 - \upsilon z\tau_1$, where x is an unbounded longitudinal coordinate, and y and z are the coordinates in the plane of a cross-section of the rod. The components of the strain tensor have the form

$$\varepsilon_{11} = \tau_1 + \tau_1^2/2, \quad \varepsilon_{12} = -\nu y\partial_x\tau_1/2, \quad \varepsilon_{13} = -\nu z\partial_x\tau_1/2, \quad \varepsilon_{23} = 0 \tag{1.3}$$

The expression for the elastic potential in terms of the gradient of the longitudinal displacement of points of rod (up to cubic terms inclusive) follow from relations (1.1)–(1.3)

$$\Phi = E\tau_1^2/2 + \beta\tau_1^3/3 + \mu\nu^2(y^2 + z^2)(\partial_x\tau_1)^2/2 \tag{1.4}$$

where E is Young's modulus. We shall henceforth assume that the quantity

$$\beta = 3E/2 + A(1 - 2\nu^3) + 3B(1 - 2\nu + 2\nu^2 - 4\nu^3) + C(1 - 2\nu)^3$$

(which is negative for the majority of solids[23]) is less than zero.

The linear kinetic energy density K of points of the rod has the form

$$K = \rho S(\partial_t u)^2/2 + \rho\nu^2 J(\partial_{tx}^2 u)^2/2; \quad J = \iint\limits_S (y^2 + z^2)dydz \tag{1.5}$$

where ρ is the rod density, S is the cross-section area and J is the polar moment of inertia.

It follows from relations (1.4) and (1.5) that the Lagrangian is given by the formula

$$L = K - \iint\limits_S \Phi dydz =$$

$$= \rho S(\partial_t u)^2/2 + \rho\nu^2 J(\partial_{tx}^2 u)^2/2 - ES(\partial_x u)^2/2 - \beta S(\partial_x u)^3/3 - \mu\nu^2 J(\partial_{xx}^2 u)^2/2$$

Varying the action

$$\int\limits_{t_0}^{t}\int\limits_{-\infty}^{\infty} Ldtdx$$

we obtain an equation describing the long weakly non-linear longitudinal waves in the elastic rod

$$\rho S\partial_{tt}^2 u = ES\partial_x\tau_1 + \rho\nu^2 J\partial_{ttx}^3\tau_1 - \mu\nu^2 J\partial_{xxx}^3\tau_1 + \beta S\partial_x\tau_1^2 \tag{1.6}$$

Differentiating both sides of Eq. (1.6) with respect to x, carrying out the scale transformations

$$x \to \nu R_i x, \quad t \to \frac{\nu R_i}{c_s}t, \quad \tau_1 \to \frac{E}{|\beta|}\tau_1; \quad R_i = \sqrt{\frac{J}{S}}, \quad c_s = \sqrt{\frac{E}{\rho}} \tag{1.7}$$

where R_i is the radius of inertia of the rod and c_s is the speed of the rod, and retaining the old notation for the variables, we obtain

$$\partial_{tt}^2(1-\partial_{xx}^2)\tau_1 = \partial_{xx}^2(1-R_s^2\partial_{xx}^2)\tau_1 - \partial_{xx}^2\tau_1^2; \quad R_s = \sqrt{\mu/E} \tag{1.8}$$

where R_s of the shear rate and the speed of the rod. In the case of practical materials $R_s^2 < 1$ (for example, $R_s^2 \approx 0, 4$ for steel).
The equation of travelling waves, which depend on the self-similar variable $\xi = x - ct$ and decay at infinity, has the form

$$(R_s^2 - c^2)\partial_{\xi\xi}^2\tau_1 = (c^2-1)\tau_1 + \tau_1^2 \tag{1.9}$$

and admits of the soliton solutions

$$\tau_1^0 = -\frac{3}{2}(c^2-1)\,\mathrm{ch}^{-2}\frac{1}{2}\sqrt{\frac{c^2-1}{c^2-R_s^2}}\xi \tag{1.10}$$

Solutions (1.10) describe solitons of two families: 1) for $c^2 > 1$ and 2) for $c^2 < R_s^2$. The solitons of the first family are compression waves ($\tau_1 < 0$), and those of the second family are tension waves ($\tau_1 > 0$). The family of tension solitons consists of waves of finite amplitude, the carrier of which tends to zero when $c^2 \to R_s^2$ and, at the same time, the amplitude remains bounded. Note that Eq. (1.8) was derived for long waves of small amplitude (that is, for solitons $c^2 - 1 \ll 1$) and the family of tension solitons is the solution of Eq. (1.9), describing waves which do not correspond to physical processes for which the initial Eq. (1.8) was derived, and the tension solitons will therefore be excluded from consideration.

Generally speaking, the existence of the above-mentioned family among the solutions of Eqs. (1.8) is due to the presence of “competing dispersions” in the model of the rod considered: one of them is the result of the radial motion of the rod and the other is the result of the existence of shear strains. For wave speeds close to unity, the shear effects are insignificant and, without any loss of generality, we shall therefore henceforth assume that $R_s = 0$. Equation (1.8) then takes the form of the Boussinesq equation

$$\partial_{tt}^2(1-\partial_{xx}^2)\tau_1 = \partial_{xx}^2\tau_1 - \partial_{xx}^2\tau_1^2 \tag{1.11}$$

and the unique family of solutions of this equation, describing compression solitons, has the form

$$\tau_1^0 = -\frac{3}{2}(c^2-1)\,\mathrm{ch}^{-2}\frac{\sqrt{1-c^{-2}}\xi}{2} \tag{1.12}$$

Equation (1.11) is written in the Hamiltonian form (υ_1 is the speed of a point of the rod)

$$\partial_t \mathbf{w} = \mathcal{J}\delta H/\delta\mathbf{w}; \quad \mathbf{w} = \{\tau_1, \upsilon_1\}^T \tag{1.13}$$

provided that

$$\partial_t\tau_1 = \partial_x\upsilon_1 \tag{1.14}$$

where H is the Hamiltonian and $\mathcal{J}$ is a skew-symmetric operator, which are defined by the expressions

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\left(\upsilon_1^2 + (\partial_t\tau_1)^2 + \tau_1^2 - \frac{2}{3}\tau_1^3\right)dx, \quad \mathcal{J} = \left\|\begin{matrix} 0 & (1-\partial_x^2)^{-1} \\ (1-\partial_x^2)^{-1} & 0 \end{matrix}\right\|\partial_x \tag{1.15}$$

and δ denotes a variational derivative.
By virtue of translational invariance, Eqs. (1.13)–(1.15) allow of further conserved quantity:

$$Q = \int_{-\infty}^{\infty}(\tau_1\upsilon_1 + \partial_x\tau_1\partial_x\upsilon_1)dx$$

Note that the equations, which are satisfied by the soliton solutions (1.12), can be rewritten in the form

$$\frac{\delta F}{\delta\mathbf{w}}(\phi_c) = 0; \quad F(\mathbf{w}) = H(\mathbf{w}) + cQ(\mathbf{w}), \quad \phi_c = \{\tau_1^0, \upsilon_1^0\}^T, \quad \upsilon_1^0 = -c\tau_1^0$$

The solitons (1.12) do not furnish a conditional minimum to the functional $F(w)$ and, on account of this, it is difficult to prove their Lyapunov stability (see Ref. 24, for example). Nevertheless, spectral stability holds for solitons of sufficiently small amplitude (and Eqs. (1.11) were precisely derived for waves of sufficiently small amplitude), that is, these solitons are linearly exponentially stable in the accompanying reference system.

We will write Eq. (1.11) in the Galilean reference system accompanying the soliton which moves at a speed c and we also make the substitution $\tau_1 \to c^2\tau_1$, $t \to t/c$, retaining the old notation. We obtain

$$\partial_{tt}^2\tau_1 - 2\partial_{t\xi}^2\tau_1 + \varepsilon\partial_{\xi\xi}^2\tau_1 - \partial_{tt\xi\xi}^4\tau_1 + 2\partial_{t\xi\xi\xi}^4\tau_1 - \partial_{\xi\xi\xi\xi}^4\tau_1 + \partial_{\xi\xi}^2\tau_1^2 = 0; \quad \varepsilon = 1 - c^{-2} \tag{1.16}$$

The expression for the family of solitons (1.12) in the new variables is rewritten in the form

$$\tau_1^0 = -\frac{3}{2}\varepsilon\,\mathrm{ch}^{-2}\frac{\varepsilon^{1/2}\xi}{2} \tag{1.17}$$

On making the substitution

$$\tau_1 = \tau_1^0 + w(\xi)\exp(\lambda t) \tag{1.18}$$

in Eq. (1.16), where λ is a spectral parameter and linearizing the resulting equation with respect to the soliton solution (1.17), we obtain an equation which is a generalized eigenvalue problem

$$\lambda^2 w - 2\lambda\partial_\xi w + \varepsilon\partial_{\xi\xi}^2 w - \lambda^2\partial_{\xi\xi}^2 w + 2\lambda\partial_{\xi\xi\xi}^3 w - \partial_{\xi\xi\xi\xi}^4 w + 2\partial_{\xi\xi}^2(\tau_1^0 w) = 0 \tag{1.19}$$

A soliton is said to be spectrally unstable if a solution of problem (1.19) exists with an exponentially decreasing function $w(\xi)$ and $\mathrm{Re}\lambda > 0$ at both infinities. Otherwise, a soliton is spectrally stable.

Waves of the form $\tau_1(\xi,t) = \varepsilon\tau^*(\zeta,T)$, where

$$\zeta = \varepsilon^{1/2}\xi, \quad T = \varepsilon^{3/2}t \tag{1.20}$$

are described, apart from terms $O(\varepsilon)$ for small ε, by the Korteweg–de Vries (KdV) equation

$$2\partial_T\tau_* - \partial_\zeta\tau_* + \partial_{\zeta\zeta\zeta}^3\tau_* - \partial_\zeta\tau_*^2 = 0 \tag{1.21}$$

and the soliton

$$\tau_*^0(\zeta) = \varepsilon^{-1}\tau_1^0 = -\frac{3}{2}\mathrm{ch}^{-2}\frac{\zeta}{2}$$

satisfies this equation exactly.

We define $\Lambda = \varepsilon^{3/2}\lambda$ in accordance with relation (1.20). Problem (1.19) is then written in the equivalent form

$$\partial_\zeta[2\Lambda w_* - \partial_\zeta w_* + \partial_{\zeta\zeta\zeta}^3 w_* + 2\partial_\zeta(\tau_*^0 w_*)] = \varepsilon[2\Lambda\partial_{\zeta\zeta\zeta}^3 + \Lambda^2(1 - \varepsilon\partial_{\zeta\zeta}^2)]w_*; \; w_*(\zeta) = w(\xi) \tag{1.22}$$

The derivative of the left-hand side of the KdV equation (1.21), linearized on the background of the soliton

$$2\Lambda w_* - \partial_\zeta w_* + \partial_{\zeta\zeta\zeta}^3 w_* + 2\partial_\zeta(\tau_*^0 w_*) = 0 \tag{1.23}$$

occurs on the left-hand side of equality (1.22).

It has been established[3] that, for sufficiently small ε, the eigenvalues of problem (1.22) in domains containing a closed right complex half-plane are close to the eigenvalues of problem (1.23), for which it is known that the unique eigenvalue in the right complex half-plane is a zero of multiplicity two, and it has been proved[3] that the spectrum of problem (1.22) is also not contained in the right complex half-plane, with the exception of the double zero eigenvalue. This indicates the spectral stability of the solitons (1.12) for speeds c which are close to unity. A similar spectral stability with neutral eigenvalues, equal to zero, which is referred to as strong stability,[3] plays a central role in the proof of the asymptotic stability of soliton solutions in other systems.[25]

2. Longitudinal-flexural waves

In taking account of the long transverse waves of small amplitude in the Rayleigh model, terms are added to the right-hand side of the expression for the longitudinal strain ε_{11} (1.3) which are due to he transverse displacements of the rod:

$$\varepsilon_{11} = -z\partial_x\tau_2 + \tau_1 + (\tau_1^2 + \tau_2^2)^2/2; \quad \tau_1 = \partial_x u, \quad \tau_2 = \partial_x \upsilon$$

where $\upsilon = \upsilon(x,t)$ is the transverse displacement of a point of the rod. The kinetic energy and Lagrangian, to which is added the flexural energy in the lowest order of smallness with respect to the amplitude of the wave, have the form

$$K = \rho[S(\partial_t u)^2 + S(\partial_t\upsilon)^2 + J_1(\partial_{tx}^2\upsilon)^2 + \nu^2 J(\partial_{tx}^2 u)^2]/2; \quad J_1 = \iint_S z^2 ds$$

$$L = K - J_1 E(\partial_{xx}^2\upsilon)^2/2 - ES[(\partial_x u)^2 - \partial_x u(\partial_x\upsilon)^2]/2 - \beta S(\partial_x u)^3/3 - \mu\nu^2 J(\partial_{xx}^2 u)^2/2 \tag{2.1}$$

The condition that the cubic terms in the expression for the strain potential energy are of the same order as the dispersion terms, representing the contribution to the Lagrangian from the shear and flexural potential energy, imposes a constraint on the relation between the amplitudes and wavelengths of the waves being considered. Suppose the transverse size of the rod is fixed and ε is the characteristic magnitude of the deformations. Then, $a_w/l_w \sim \varepsilon^{3/2}$, where a_w and l_w are the characteristic amplitude and wavelength respectively.

We write the equations obtained from Lagrangian (2.1) after transformations (1.7) and $\tau_2 \to \sqrt{2E/|\beta|}\tau_2$ in the form (the old notation is retained)

$$\begin{aligned}\partial_{tt}^2\tau_1 &= \partial_{xx}^2\tau_1 + \partial_{ttxx}^4\tau_1 - R_s^2\partial_{xxxx}^4\tau_1 - \partial_{xx}^2\tau_1^2 + \partial_{xx}^2\tau_2^2\\ \partial_{tt}^2\tau_2 &= -\gamma^{-2}\partial_{xxxx}^4\tau_2 + \gamma^{-2}\partial_{ttxx}^4\tau_2 + l\partial_{xx}^2\tau_1\tau_2; \quad \gamma = \nu\sqrt{J/J_1}, \quad l = E/|\beta|\end{aligned} \tag{2.2}$$

Guided by the considerations set out in Section 1, we put $R_s = 0$, in Eqs. (2.2) we transfer to a system moving with a speed c and carry out the transformation

$$\tau_i \to c^2\tau_i, \quad i = 1, 2, \quad t \to t/c$$

As a result, we obtain

$$(\mathcal{D}_\varepsilon - \mathcal{D}_1\partial_{\xi\xi}^2)\tau_1 + \partial_{\xi\xi}^2(\tau_1^2 - \tau_2^2) = 0, \quad \mathcal{D}\tau_2 - l\partial_{\xi\xi}^2(\tau_1\tau_2) = 0 \tag{2.3}$$

where

$$\begin{aligned}&\mathcal{D}_\varepsilon = \partial_{tt}^2 - 2\partial_{t\xi}^2 + \varepsilon\partial_{\xi\xi}^2, \quad \mathcal{D}_1 = \partial_{tt}^2 - 2\partial_{t\xi}^2 + \partial_{\xi\xi}^2, \quad \mathcal{D} = (1-\gamma^{-2})\mathcal{D}_1 + \gamma^{-2}c^{-2}\partial_{\xi\xi\xi\xi}^4,\\ &\xi = x - t, \quad \varepsilon = 1 - c^{-2}\end{aligned}$$

Linearizing Eq. (2.3) close to the solutions τ_1^0 and τ_2^0, where τ_3^0 is given by expression (1.7) and $\tau_2^0 = 0$, and putting

$$\begin{Vmatrix}\tau_1\\ \tau_2\end{Vmatrix} = \begin{Vmatrix}\tau_1^0 + \delta\tau_1\\ \delta\tau_2\end{Vmatrix}$$

we obtain two independent linear equations, the first of which, after the substitution (1.18), reduces to Eq. (1.19), while the second equation has the form

$$\mathcal{D}\delta\tau_2 - l\partial_{\xi\xi}^2(\tau_1^0\delta\tau_2) = 0 \tag{2.4}$$

Equation (2.4) describes the evolution of the linear flexural perturbations of the longitudinal soliton (1.17) which, as was indicated in Section 1, is linearly stable with respect to longitudinal perturbations for sufficiently small ε. Only flexural modes, with a group velocity corresponding to the speed of the soliton, can destabilize the soliton.

On the basis of this, we make the substitution

$$\delta\tau_2(\xi, t) = \tau(\xi, t)\exp[ik_0(\xi - Vt)] \tag{2.5}$$

in Eq. (2.4), where k_0 and V correspond to a mode, the group velocity of which is equal to unity and, for small ε, differs slightly from the speed c of the soliton. These quantities are determined from the system of equations

$$(V+1)^2 = 1 - \frac{1}{1+\kappa_0^2}, \quad V = -\frac{1}{(1+\kappa_0^2)^2}, \quad \kappa_0 = \gamma^{-1}k_0$$

We have

$$\kappa_0 = \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.786, \quad V = \frac{1-\sqrt{5}}{1+\sqrt{5}} \approx -0.382$$

We shall seek $\tau(\xi,t)$ in equality (2.5) in the form of the eigenfunction

$$\tau(\xi, t) = w(\xi)\exp(i\alpha t) \tag{2.6}$$

Substituting expressions (2.5) and (2.6) into Eq. (2.4), we obtain a fourth order ordinary differential equation in the function $w(\xi)$. We subsequently put $w(\xi) = w*(\zeta)$, where, as previously, $\zeta = \varepsilon^{1/2}\xi$ and seek the solution of this equation in the form of the asymptotic series

$$\begin{aligned}\alpha &= \varepsilon\alpha_0 + i\varepsilon^{3/2}\alpha_1 + \varepsilon^2\alpha_2 + i\varepsilon^{5/2}\alpha_3 + \dots\\ w_*(\xi) &= w_*^{(0)}(\zeta) + i\varepsilon^{1/2}w_*^{(1)}(\zeta) + \varepsilon w_*^{(2)}(\zeta) + i\varepsilon^{3/2}w_*^{(3)}(\zeta)\dots\end{aligned} \tag{2.7}$$

Substituting series (2.7) into the equation which $w(\xi)$ obeys, we obtain that the coefficient of $\varepsilon^{1/2}$ is equal to zero and, equating the expressions accompanying ε, we obtain the equation

$$\mathcal{L}_0 w_*^{(0)} = 0; \quad \mathcal{L}_0 = 2\alpha_0\gamma\kappa_0(1+V)(1+\kappa_0^2) - \frac{\gamma^2\kappa_0^2}{4} + \gamma^2\kappa_0^2 l\tau_0^* + $$
$$+ (1 + 6\kappa_0^2 V + \kappa_0 V^2)\frac{d^2}{d\zeta^2} \tag{2.8}$$

and, in the case of powers of ε greater than unity, an infinite hierarchy of equations:

$$\mathcal{L}_0 w_*^{(i)} + 2\alpha_i\gamma\kappa_0(1+V)(1+\kappa_0^2)w_*^{(0)} = F_i(\zeta, w_*^{(j)}, \alpha_j), \quad 0 \le j < i \ge 1 \tag{2.9}$$

For $i = 1$, for example, the expression for the right-hand side of F_1 is given by the formula

$$F_1 = -2\left[\alpha_0(1+2V)(1-\kappa_0^2) + \frac{\gamma\kappa_0^2}{2}\right]\frac{dw_*^{(0)}}{d\zeta} \tag{2.10}$$

The operator $\mathcal{L}_0$ is self-adjoint, the coefficient of the second derivative $1 + \kappa V(6\kappa_0 + V) \approx -0.301 < 0$ and, therefore, its discrete spectrum is not empty and is determined by the equality

$$(\alpha_0)_n = \frac{\gamma}{2l}\frac{\kappa_0^2}{1+\kappa_0^2} - \frac{1+\kappa_0^2(6+V)}{96\gamma\kappa_0^2 l(1+\kappa_0^2)}\left[(1-2n) + \sqrt{1 - \frac{24\gamma\kappa_0^2 l}{1+\kappa_0^2(6+V)}}\right]^2$$

$$n = 0, 1, 2, \ldots, N < \frac{1}{2}\left(1 + \sqrt{1 - \frac{24\gamma\kappa_0^2 l}{1+\kappa_0^2(6+V)}}\right)$$

and the eigenfunctions $w_{*n}^{(0)}$ are expressed in terms of hypergeometric functions.[26] Moreover, the eigenfunctions with the number $n = 0$, 2, 4, ... are even and those with the numbers $n = 1, 3, 5, \ldots$ are odd.

A second linearly independent solution of the equation $\mathcal{L}_0 w_{**}^0 = 0$ has the form

$$w_{**}^{(0)} = aw_*^{(0)} + bw_*^{(0)}\int (w_*^{(0)})^{-2} d\zeta; \quad a = \text{const}, \quad b = \text{const} \tag{2.11}$$

and a parity opposite to the parity of $w_*^{(0)}$ if the constants a and b in relation (2.11) are determined from the condition that the Wronskians $w_*^{(0)}$ and $w_{**}^{(0)}$ are equal to unity.

The compatibility conditions for Eq. (2.9)

$$\alpha_i = r^{-1}\frac{\langle F_i, w_*^{(0)}\rangle}{\langle w_*^{(0)}, w_*^{(0)}\rangle}; \quad r = 2\gamma\kappa_0(1+V)(1+\kappa_0^2)$$

where $\langle\cdot,\cdot\rangle$ is a scalar product in $L_2(\mathbb{R})$, give $\alpha_1 = 0$ by virtue of the fact that F_1 has a parity which is opposite to that of $w_*^{(0)}$, as follows from equality (2.10). Furthermore, from the compatibility condition on any step n, we have $\alpha_i = 0$ for $i < n$ and odd i since F_i has a parity for the corresponding i which is opposite to that of $w_*^{(0)}$ and the parity of F_i for even i is the same as that of $w_*^{(0)}$.

The solution of Eq. (2.9) for the corrections $w_*^{(i)}$ has the form

$$w_*^{(i)} = w_{**}^{(0)}\int_\zeta^\infty w_*^{(0)}(F_i - r\alpha_i w_*^{(0)})d\zeta + w_*^{(0)}\int_0^\zeta w_{**}^{(0)}(F_i - r\alpha_i w_*^{(0)})d\zeta$$

and is a function of the same parity as $w_*^{(0)}$ for even i and opposite parity in the case of odd i.

Hence, for $n \ge 1$, the values α_{2n-1}, which are the imaginary corrections for the eigenvalue α in representation (2.6), are equal to zero and the compression soliton considered is neutrally stable.

3. Discussion

Hence, we have analysed the stability of moving domains of compression, that is, solitons with respect to flexural perturbations in isotropic non-linear elastic rods. The interaction of the flexural and longitudinal modes in such rods is described by the cubic powers of the gradients of the displacements in the elastic potential. The higher powers of the gradients of the displacements are assumed to have a higher order of smallness and are not taken into consideration. It is also postulated that the flexural energy is proportional to the square of the second derivatives of the vertical displacement, which is also due to the relative smallness of the terms of the following order. These two assumptions mean that the model considered can be used to describe waves of fairly small amplitude and having a long wavelength compared with the transverse dimensions of the rod.

The equations of this model have the form of (2.2). It follows from the first equation that a constant compression with $c\ \tau_1^c = -P$, $p =$ const > 0 is permissible for the rod. From the second equation, it follows, in particular, that linear flexural perturbations $\delta\tau_2$ on a background of constant compression obey the equation

$$\partial_{tt}^2\delta\tau_2 = \gamma^{-2}(\partial_{tt}^2 - \partial_{xx}^2 - lP)\partial_{xx}^2\delta\tau_2$$

Putting

$$\delta\tau_2 \sim \exp(\alpha t - ikx)$$

we obtain

$$\alpha = \pm\sqrt{\frac{k^2(lP - \gamma^{-2}k^2)}{1 + \gamma^{-2}k^2}}$$

whence it follows that long wave perturbations increase exponentially for any $P > 0$, which implies that a uniformly compressed rod is unstable under flexural for any compression which may be as small as desired.

We have established that, in the case of a uniform motion of compression domains, representing the propagation of a solitary wave, described by a soliton solution of the model equations (1.8) and (2.2), the rod is found to be neutrally stable towards flexure. In this case, the stability is explained by two factors: firstly, a compression wave has a velocity which exceeds a certain value c_s, which turns out to be sufficient for stabilization and, secondly, the intensity of the compression decreases in the wave when it recedes to infinity which, apparently, is also a stabilizing action.

The stability of solitons can be successfully analysed asymptotically due to the presence in the problem of a small parameter ε, characterizing the magnitude of the amplitude of the waves. The stability problem reduces to finding the discrete spectrum of the linear self-adjoint operator $\mathcal{L}_0$ in Eq. (2.8) and recovering the following terms of the asymptotic series for the eigenfunction $w*(\zeta)$ and the eigenvalue α from Eq. (2.9). In this case, it is found that it follows from the symmetry properties of the compression wave that all the imaginary terms in the expansion of α with respect to the small parameter ε are identically equal to zero, which implies that the compression soliton is neutrally stable.

It is obvious that a flexural perturbation which interacts most strongly with a compression wave must propagate with the group velocity at least in the zeroth order with respect to ε, which is identical with the propagation velocity of the compression wave. This imposes a constraint on the length of the desired flexural perturbation L_b which, as follows from relations (1.7) and (2.5), is given by the expression

$$L_b = \frac{2\pi\nu R_i}{k_0} = 2\frac{\pi}{\kappa_0}R_i\frac{J_1}{J}$$

In the case of a beam of rectangular cross-section $c \times d$, we have

$$R_i = \sqrt{\frac{c^2 + d^2}{3}}, \quad L_b = \frac{2\pi}{\kappa_0\sqrt{3}}d \approx 4.62d$$

According to the magnitude of the ratio of the wavelength of the flexural wave to the beam thickness, the Rayleigh model of a rod used for a beam of rectangular cross-section describes flexural waves trapped by a compression wave quite well. In the case of a rod of circular cross-section, similar estimates show that the wavelength of the trapped wave is roughly equal to two diameters of the rod. Flexural waves of this wavelength are poorly described by the Rayleigh model, and, to describe them, it is necessary to use a more precise model when analysing of the stability of compression solitons and, in particular, to take account of the dilatation of the middle line of the rod and the non-conservation of the orthogonality of the middle line in the normal sections accompanying flexure. Analysis of stability using the more precise model apparently leads to the same conclusions by virtue of the preservation of the symmetry properties of the eigenfunctions of a sequence of eigenvalue problems similar to (2.9), which are due to the space-time reversibility of the initial equations.

Acknowledgements

I wish to thank A. G. Kulikovskii for discussion and I. B. Bakholdin for remarks.

The research was financed by the Russian Foundation for Basic Research (05-01-00219) and the Programme for the Support of Leading Scientific Schools (NSh-4710.2006.1).

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Translated by E.L.S.